$$S=4n\int_0^b\int_0^{mx}\sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2}\,dxdy=4n\int_0^b\int_0^{mx}\frac{Rdxdy}{\sqrt{R^2-x^2-y^2}}$$

$$=4nR\int_0^b\sin^{-1}\frac{y}{\sqrt{R^2-x^2}}\Big]_0^{mx}dx=4nR\int_0^b\sin^{-1}\left(\frac{mx}{\sqrt{R^2-x^2}}\right)dx,$$

$$=4nR\left[x\sin^{-1}\frac{mx}{\sqrt{R^2-x^2}}-mR\int\frac{xdx}{(R^2-x^2)\sqrt{R^2-(1+m^2)x^2}}\right]_0^b,$$

by integrating by parts.

To integrate the last expression, let $x=R\sin\theta/\sqrt{1+m^2}$. The integral then becomes

$$-mR\int\frac{\sin\theta d\theta}{1+m^2-\sin^2\theta}=mR\int\frac{-\sin\theta d\theta}{m^2+\cos^2\theta}=R\tan^{-1}\left(\frac{\cos\theta}{m}\right)=R\tan^{-1}\left[\frac{\sqrt{R^2-(1+m^2)x^2}}{mR}\right].$$

Hence,

$$S=4nR\left[x\sin^{-1}\frac{mx}{\sqrt{R^2-x^2}}+R\tan^{-1}\frac{\sqrt{R^2-(1+m^2)x^2}}{mR}\right]_0^b$$

$$=4nR\left\{b\sin^{-1}\left(\frac{mb}{\sqrt{R^2-b^2}}\right)-R\left[\tan^{-1}\left(\frac{1}{m}\right)-\tan^{-1}\frac{\sqrt{R^2-(1+m^2)b^2}}{\sqrt{R^2-b^2}}\right]\right\},$$

$$=4nR\left\{b\sin^{-1}\frac{mb}{\sqrt{R^2-b^2}}-R\sin^{-1}\left[\frac{m}{1+m^2}\left(\frac{R-\sqrt{R^2-(1+m^2)b^2}}{\sqrt{R^2-b^2}}\right)\right]\right\},$$

$$=4nR\left\{b\sin^{-1}\tan\left(\frac{\pi}{n}\cdot\frac{b}{\sqrt{R^2-b^2}}\right)-R\sin^{-1}\left[\tfrac{1}{2}\sin\frac{2\pi}{n}\cdot\frac{R-\sqrt{R^2-b^2\sec^2\frac{\pi}{n}}}{\sqrt{R^2-b^2}}\right]\right\}.$$

When $n=4$,

$$S=8R\left[2b\sin^{-1}\frac{b}{\sqrt{R^2-b^2}}-2R\sin^{-1}\left(\frac{1}{2}\cdot\frac{R-\sqrt{R^2-2b^2}}{\sqrt{R^2-b^2}}\right)\right],$$

$$=8R\left[2b\sin^{-1}\frac{b}{\sqrt{R^2-b^2}}-R\sin^{-1}\left(\frac{b^2}{R^2-b^2}\right)\right],$$

a result agreeing with that in Osborne's *An Elementary Treatise on the Differential and Integral Calculus*, p. 270, example 5. When $n\doteq\infty$, the prism $\doteq$ a cylinder and

$$S\doteq 4R\left[\frac{\pi b^2}{\sqrt{R^2-b^2}}-\pi R\left(\frac{R-\sqrt{R^2-b^2}}{\sqrt{R^2-b^2}}\right)\right]=4\pi R\left[\frac{b^2-R^2}{\sqrt{R^2-b^2}}+R\right]=4\pi R(R-\sqrt{R^2-b^2}),$$

that is, twice the circumference of a great circle of the sphere times the altitude of the zone, or the surface of two zones of altitude $(R-\sqrt{R^2-b^2})$.

Also solved by J. W. CLAWSON.

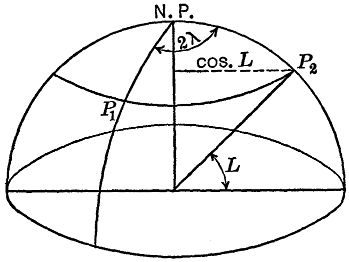
381. Proposed by ELBERT H. CLARK, Purdue University.

Of all points having the same latitude and a constant difference α in their longitudes, to find the latitude of the two so situated that the distance between them, measured along their common parallel of latitude, shall exceed the distance between them measured on their great circle by the greatest possible amount.

SOLUTION BY W. C. EELLS, U. S. Naval Academy.

Let L be latitude of the points, 2λ (in radians) their difference in longitude, $2d$ and $2D$ the small circle latitude and great circle distances between them, and $y=d-D$. Let the earth be a sphere of radius unity. Since the radius of the latitude circle is $\cos L$,

$$d=\lambda\cos L. \qquad (1)$$



Dropping a perpendicular from the north pole to the great circle through the two points, we have two right triangles. Whence, by Napier's rules of circular parts,

$$\sin D = \sin \lambda \cos L. \tag{2}$$

From (1) and (2), $y = \lambda \cos L - \text{arc sin } (\sin \lambda \cos L)$, and it is desired to find the maximum value of y.

Let

$$\frac{dy}{dL} = -\lambda \sin L + \frac{\sin L \sin h}{\sqrt{1 - \sin^2 \lambda \cos^2 L}} = 0.$$

Then, $\sin L = 0$ is a solution, giving $L = 0$, obviously the minimum value of y.

The other solution is

$$\cos L = \pm \frac{\sqrt{\lambda^2 - \sin^2 \lambda}}{\lambda \sin \lambda},$$

the maximum. Whence, L can be found for any given λ. As λ varies from 0° to 90°, 2λ varies from 0° to 180° and L varies from 90° to 39° 32′ 23″. For $\lambda = 30°$, $L = 53° 34' 42''$ and for $\lambda = 60°$, $L = 49° 31' 16''$.

Also solved by J. A. Caparo, Paul Capron, Horace Olson, C. N. Schmall.

382. Proposed by B. J. BROWN, Student in Drury College.

Discuss for what values of m and n, the integral, $\int_0^1 x^{m-1}(1-x)^{n-1}dx$, is finite and show how this integral can be expressed by means of integrals of the form $\int_0^\infty e^{-x}x^{p-1}dx$.

Solution by H. L. Agard, Williams College.

1°. The integrand has no singular points for $m \geqq 1$ and $n \geqq 1$. The only singular points are $x = 0$ for $m < 1$ and $x = 1$ for $n < 1$.

Pierpont[1] states the

Theorem. Let $f(x)$ be regular in the interval (a, b), except at a. For some $0 < \mu < 1$, let $R \lim_{x=a} (x-a)^\mu |f(x)|$ be finite. Then $f(x)$ is absolutely integrable in (a, b).

[The symbol $R \lim_{x=a}$ indicates that $x \doteq a$ from the right. A similar meaning attaches to $L \lim_{x=a}$. The theorem applies equally to a singularity at b.]

Consider

$$R \lim_{x=0} x^\mu |x^{m-1}(1-x)^{n-1}|, \quad m < 1.$$

Evidently, this limit is finite for $m > 0$ and $1 - m < \mu < 1$.

Consider

$$L \lim_{x=1} (1-x)^\mu |x^{m-1}(1-x)^{n-1}|, \quad n < 1.$$

Evidently, this limit is finite for $n > 0$ and $1 - n < \mu < 1$.

Hence, by the above theorem, the integral is absolutely convergent if both $m > 0$ and $n > 0$.

[1] Pierpont, *Theory of Functions of Real Variables*, Vol. I, p. 407.